USEFUL THEOREMS AND FORMULAS FROM CALC:

Special Limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Theorems About Continuity and Function Values:

Intermediate Value Theorem: If a function $f(x)$ is continuous on $[a, b]$, then the function will take on all the values of y between $f(a)$ and $f(b)$.

Continuity of Composite Functions: If g is continuous at c and f is continuous at $g(c)$, then $(f \circ g)(c)$ is continuous at c.

Derivatives and Differentiability:

Definition of the derivative: $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

Common Derivative Rules:

$f(x)$	$f'(x)$
c	0
x^n	nx^{n-1}
$c \cdot f(x)$	$c \cdot f'(x)$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$f(x) \cdot g(x)$	$f(x) \cdot g'(x) + f'(x) \cdot g(x)$
$\frac{f(x)}{g(x)}$	$\frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
e^x	e^x
$\ln x$	$1/x$

☆ If $f(x)$ is differentiable at $x = c$, then $f(x)$ is also continuous at $x = c$.

First Derivative Test:

- If $f'(x) > 0$, then $f(x)$ is increasing.
- If $f'(x) < 0$, then $f(x)$ is decreasing.
- If $f'(x) = 0$, then $f(x)$ is constant.

Extreme Value Theorem: If f is continuous on a closed interval $[a,b]$, then f has both a minimum and a maximum on the interval.

Second Derivative Test:

- If $f''(x) > 0$, then $f(x)$ is concave up.
- If $f''(x) < 0$, then $f(x)$ is concave down.
- If $f''(x) = 0$, the test cannot be used.

Theorems with Derivative Applications:

Rolle's Theorem:

If $f(x)$ is continuous on the interval $[a,b]$ and differentiable on (a,b), and $f(a) = f(b)$, then there is a number c such that $a < c < b$, and $f'(c) = 0$.

Mean Value Theorem (for derivatives):

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is a number c such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Derivatives of More Complex Functions:

$f(x)$	$f'(x)$
a^u	$u' a^u \ln a$
$\log_a u$	$\frac{u'}{u \ln a}$
$\arcsin u$	$\frac{u'}{\sqrt{1-u^2}}$
$\arccos u$	$\frac{-u'}{\sqrt{1-u^2}}$
$\arctan u$	$\frac{u'}{1+u^2}$

$f(x)$	$f'(x)$
$\text{arccot}\, u$	$\frac{-u'}{1+u^2}$
$\text{arcsec}\, u$	$\frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$
$\text{arccsc}\, u$	$\frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$

Methods of Approximation:

Newton's Method for Approximating Zeros of a Function:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

⇒ Rule for Finding the n^{th} Root of a:

$$x_{q+1} = \frac{1}{n}\left(x_q(n-1) + \frac{a}{x^{n-1}}\right)$$

Tangent Line Approximation for Function Values:

$$dy = f'(x) \cdot dx$$

Euler's Method for Approximating Function Values:

$$x_1 = x_0 + h \qquad y_1 = y_0 + h(y'(x_0, y_0))$$

Rectangular Approximation Methods for Definite Integrals $\left(\int_a^b f(x)\,dx\right)$:

$$LRAM = \frac{b-a}{n}\sum_{i=1}^{n} f(l_i) \qquad MRAM = \frac{b-a}{n}\sum_{i=1}^{n} f(m_i)$$

$$RRAM = \frac{b-a}{n}\sum_{i=1}^{n} f(r_i)$$

Trapezoidal Approximation for Definite Integrals:

$$TRAP = \frac{b-a}{2n}\left(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right)$$

Simpson's Rule for Approximating Definite Integrals:

$$SIMP = \frac{b-a}{3n}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 4f(x_{n-1}) + f(x_n)\right)$$

Theorems with Integral Applications:

FUNDAMENTAL THEOREM OF CALCULUS:

1. If f is nonnegative and continuous on [a, b], then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

2. $$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Mean Value Theorem (for integrals):

If f is continuous on [a, b], there is a value of $x = c$ such that

$$f(c) = f_{AVE} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Integration by Parts:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Sequences and Series:

* If $\lim_{n\to\infty} a_n$ exists, then the sequence a_n converges.

* If a sequence is bounded and monotonic, then it converges.

Tests for Convergence/divergence of infinite series:

Geometric series: $\sum_{n=0}^{\infty} a \cdot r^n$ converges when $|r| < 1$, and diverges otherwise.

* For $|r| < 1$, $\sum_{n=0}^{\infty} a \cdot r^n = \frac{a}{1-r}$

n^{th}-Term Test: For a series $\sum_{n=0}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $0 < p \leq 1$

Telescoping Series: $\sum_{n=1}^{\infty} (b_n - b_{n+1})$ converges if $\lim_{n\to\infty} b_n$ exists and $= L$

* $S = b_1 - L$

Alternating Series: $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges if $0<a_{n+1}\le a_n$ AND $\lim_{n\to\infty} a_n = 0$

★ $|R_N| \le a_{N+1}$

Integral Test: $\sum_{n=1}^{\infty} a_n$, where $a_n = f(n) \ge 0$, converges if $\int_1^{\infty} f(x)$ converges and diverges if $\int_1^{\infty} f(x)$ diverges

★ $0 < R_N < \int_N^{\infty} f(x)\,dx$

Root Test: $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $=\infty$, and is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Ratio Test: $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ or $=\infty$, and is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Taylor Polynomials and Series:

★ If f has n derivatives at $x=c$, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

approximates f and is called the n^{th} Taylor polynomial for f at c.

★ If $c=0$, $P_n(x)$ is called the n^{th} Maclaurin polynomial for f at c.

Taylor's Theorem:

$f(x) = P_n(x) + R_n(x) \Rightarrow R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$, where z is between x and c

★ The Taylor series for $f(x)$ at c is given by:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Common Taylor Series:

- $\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$
- $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$
- $\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
- $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$
- $\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n+1}}{(2^n n!)^2 (2n+1)}$

Area and Volume with Integrals:

Area Under One Curve:

$$\int_a^b f(x)\, dx$$

Area Between Two Curves:

$$\int_a^b (f(x) - g(x))\, dx, \text{ where } f(x) > g(x)$$

Volume with One Curve:

Horizontal Axis of Rotation:

$$\pi \int_a^b (f(x))^2\, dx$$

OR

$$2\pi \int_a^b (f(y))y\, dy$$

Vertical Axis of Rotation:

$$\pi \int_a^b (f(y))^2\, dy$$

OR

$$2\pi \int_a^b x(f(x))\, dx$$

Volume with Two Curves:

Horizontal Axis of Rotation:

$$\pi \int_a^b [(f(x))^2 - (g(x))^2]\, dx$$

$$2\pi \int_a^b y(f(y) - g(y))\, dy$$

Vertical Axis of Rotation:

$$\pi \int_a^b [(f(y))^2 - (g(y))^2]\, dy$$

$$2\pi \int_a^b x(f(x) - g(x))\, dx$$